WMMB 9401 8/9/025

PARTIAL DIFFERENTIAL EQUATION

Course outline

i) Curves and Surfaces with 3d
ii) Simultaneous different Equations
iii) First Order P.d.es
 - Pfuffiar differential equations
 - simi-linear differiential equations
 - Quasi-linear difential equations.

iv) Integral surface passing through a given curve.

v Methods of Cauchy charpit jacobi in Solving P.D.Es

GENERAL INTRODUCTION

- Differential equations are equation that involves derivatives.
- Most real-life situations involves rates of change which can be represented mathematicaly by differential equation.

Ordinary differential equations contains total derivatives having just a variable i) dependent.
ii) independent.

- Partial differential eqaution on the other hand, contains partial derivatives with one dependent variable and many independent variable.

Example

$$f = f(w, x, y z)$$

We can obtain $\frac{\partial f}{\partial w}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\delta z}$

Order and Degree of a differential Equation.

- The order of D.E is the highest derivative that occur in the eauation
- The Degree of D.E is the highest power of the highest derivative that occurs in an Equation.

1	Order	Degree
i) $\frac{dy}{dx} + \frac{d^2y}{dx^2} + y = 0$	2	1
ii) $\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 + 4 = 0$	3	1
iii) $\frac{\partial y}{\partial x^2} + \frac{\partial^{\textcircled{2}} y}{\partial x^2} + \left(\frac{\partial y}{\partial x}\right)^4 = 0$	2	1
iv $\frac{\partial y}{\partial x} + \left(\frac{\partial^4 y}{\partial x^4}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^{10} = 0$	4	2.

Simultaneous differential equation of the form $\frac{dx}{P} = -\frac{dy}{Q} = -\frac{dz}{R}$

We consider first order, first degree differential eauation involving 3 variable x, y, z

Consider a Simultaneous eauation

$$P_1 dx + Q_1 dy + R_1 dz = 0 \quad \text{and}$$

$$P_2 dx + Q_2 dy + R_2 dz = 0$$

Where the coefficient $P_1 P_2$ $Q_1 Q_2$ and $R_1 R_2$ are function of x, y and z

$$P_1 dx + Q_1 dy + R_1 dz = 0 \;\text{--(i)}\; \times R_2$$

$$P_2 dx + Q_2 dy + R_2 dz = 0 \;\text{-(ii)}\; \times R_1$$

$P_1R_2dx + R_2Q_1dy + R_2R_1dz = 0$

$R_1P_2dx + R_1Q_2dy + R_1R_2dz = 0$

$(P_1R_2 - P_2R_1)dx + (Q_1R_2 - R_1Q_2)dy = 0$

Change the sign.

$(P_1R_2 - P_2R_1)dx = -(Q_1R_2 - Q_2R_1)dy$

$$\frac{dx}{Q_1R_2 - Q_2R_1} = \frac{dy}{P_1R_2 - P_2R_1}$$

likewise we can eliminate dx or dy to eventually give.

$$\frac{dx}{Q_2R_1 - Q_1R_2} = \frac{dy}{P_1R_2 - P_2R_1} = \frac{dz}{P_2Q_1 - P_1Q_2}.$$

which is the form

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$ as required.

NOTE

Most Simultanous differential equation will be written (rep) in the form

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

- The nature of our Solution will be of the form :
$U_1(x, y, z) = C_1$ and $U_2(x, y, z) = C_2$
where U_1 and U_2 are independent integral while C_1 and C_2 are arbitrary Constant

Two function U_1 and U_2 are independent if $\frac{U_1}{U_2} \neq$ costant.
(One is not the Scalar Multiple of the other)
e.g if $U_1 = x^2 + y^2 + z^2$
$U_2 = x + y + z$

then

$$\frac{U_1}{U_2} = \frac{x^2 + y^2 + z^2}{x + y + z} \neq C$$

Hence U_1 and U_2 are independent but if $U_1 = x + y + z$ and
$U_2 = 2x + 2y + 2z$

then

$$\frac{U_1}{U_2} = \frac{x + y + z}{2x + 2y + 2z} = \frac{1}{2}$$

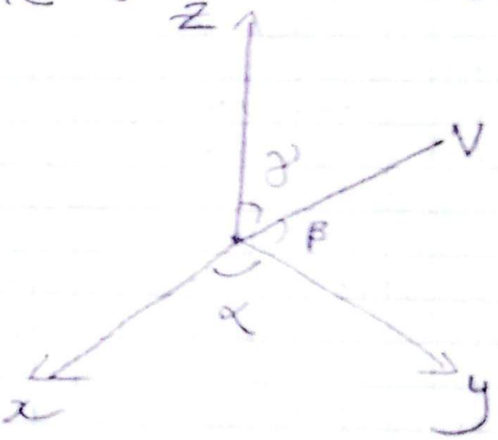
$\Rightarrow U_1$ and U_2 are dependent.

i.e $U_2 = 2U_1$

Geometric Interpretation

In 3.D Coordinate geometry the direction cosines of a tangent to a Sur are proportional to dx : dy : dz

The Constant Cosines are also proportional to P : Q : R to the Solution U_1 and U_2



Directional Cosines

$\cos \alpha = \frac{a}{|V|}$

$\cos \beta = \frac{b}{|V|}$

$\cos \gamma = \frac{c}{|V|}$

Example 1

RULE 1: Here we equate two Fraction at a time.

Example ✓

Solve $\frac{xdx}{y^2z}=\frac{dy}{xz}=\frac{dz}{y^2}$

Solution

Taking the first and the second.

$$\frac{xdx}{y^2z}=\frac{dy}{xz}$$

$$\cancel{x^2dx}\quad x^2dx=y^2dy$$

$$x^2dx-y^2dy=0$$

Intergrating.

$$\int x^2dx-\int y^2dy=\int 0.$$

$$\frac{x^3}{3}-\frac{y^3}{3}=c_1$$

Again taking first and 3rd

$$\frac{xdx}{y^2z}=\frac{dz}{y^2}$$

Rule

$$xdx=zdz$$

$$xdx-zdz=0$$

$$\int xdx-\int zdz=0.$$

$$\tfrac{1}{2}x^2-\tfrac{1}{2}z^2=c_2.$$

$$x^2-z^2=c_2\ \text{---(ii)}$$

c_2 - arbitrary Constant.

Since x^3-y^3 and x^2-z^2 are independent the general solution is given by (i) and (ii) c_1 and c_2 being arbitrary constant.

Example 2

$$\frac{dx}{yz}=\frac{dy}{zx}=\frac{dz}{xy}$$

Soln

Taking (i) and (ii)

$$\frac{dx}{yz}=\frac{dy}{zx}.$$

$$xdx=ydy$$

intergrating

$$\frac{x^2}{2}-\frac{y^2}{2}=0.$$

$$x^2-y^2=c_1\ \text{---(i)}$$

Taking (ii) and (iii)

$$\frac{dy}{zx}=\frac{dz}{xy}.$$

$$ydy=zdz.$$

$$ydy-zdz=0.$$

intergrating

$$\frac{y^2}{2}-\frac{z^2}{2}=0.$$

$$y^2-z^2=c_2\ \text{--(ii)}$$

Since (x^2-y^2) and (y^2-z^2) are independent then the general Solution is given by (i) and (ii) c_1 and c_2 being arbitrary Constant.

~~Example 3~~

~~$\frac{dx}{yz}=\frac{dy}{zx}=\frac{dz}{xy}$~~

Soln

~~Taking (i) and (ii)~~

~~$\frac{dx}{yz} = \frac{dy}{-xz}$~~

~~$xdx = ydy$~~

~~$xdx - ydy.$~~

Example 3

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}.$$

Soln

Taking the first 2

$$\frac{dx}{x} = \frac{dy}{y}.$$

$$\frac{dx}{x} - \frac{dy}{y} = 0.$$

Intergrating

$$\ln x - \ln y = \ln c_1$$

$$\ln \frac{x}{y} = \ln c_1$$

$$\frac{x}{y} = c_1$$

Taking (ii) and (iii)

$$\frac{dy}{y} - \frac{dz}{z} = 0$$

$$\ln y - \ln z = \ln c_2$$

$$\ln \frac{y}{z} = \ln c_2$$

$$\frac{y}{z} = c_2$$

Since (i) and (ii) are independent the general Soln is given by (i) and (ii) c_1 and c_2 being arbitrary constant.

Exercise

i) $\frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{x^2y^2z^2}$

ii) $\frac{dx}{x^2+ay^2} = \frac{dy}{-xy} = \frac{dz}{xz}$

iii) $\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$

iv) $dx = dy = \frac{dz}{\sin x}$

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RULE II

Considering $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ we may have only one pair of relations giving a solution $u_1(xyz) = c_1$ as discussed under rule 1 it is further possible to use $u_1(x_1, y_1, z_1) = c_1$ to obtain the next soln $u_2(x_1, y_1, z) = c_2$ by sustitution.

Example 1

Solve ~~$\frac{dx}{x-z}$~~

$$\frac{dx}{x^2(z^2+xy)} = \frac{dy}{-y^2(z^2+xy)} = \frac{dz}{x^4}$$

Solve

Taking (i) and (ii) fractions

$\frac{dx}{x} = \frac{dy}{-y}$ since $z(z^2+xy)$ is common.

or

$\frac{dx}{c} + \frac{dy}{y} = 0$ intergrating.

$\ln x + \ln y = \ln c_1$

$\ln xy = \ln c_1 = xy = c_1$ ---(i)

Using equation (i) and taking the (i) and (iii) we will have

$$\frac{dx}{x^2(z^3+c_1)} = \frac{dz}{x^4}$$

$x^3 dx = 2(z^2+c_1)dz$

$x^3 dx - (z^3+c_1 z)dz = 0$ integrating

$\frac{x^4}{4} - \frac{z^4}{4} - \frac{c_1 z^2}{2} = \frac{c_2}{4}$

$x^4 - z^4 - 2c_1 z^2 = c_2$

But $c_1 = xy$ from (i)

But $c_1 = xy$ from (i)

$x^4 - z^4 - 2xyz^2 = c_2$ ---(ii)

Since (i) and (ii) are independent then the complete soln is given by (i) and (ii) c_1 and c_2 being arbitrary constant

Example 2

Solve

$$\frac{dx}{1} = \frac{dy}{2} = \frac{dz}{5z+\tan(y-2x)}$$

Soln

Taking i) and ii)

$\frac{dx}{1} = \frac{dy}{2}$

$\frac{dx}{1} - \frac{dy}{2} = 0$

$2dx - dy = 0$. Intergrating

$2x - y = c_1$ ---(i)

Using (i) and taking (i) and (iii) we will have.

$$\frac{dx}{1} = \frac{dz}{5z+\tan c_1}$$

$x = \frac{1}{5}\ln(5t^2 + \tan c_1) + c_2$

$x = \frac{1}{5}\ln(5z + \tan(y-2x) + c_2$.

Since (i) and (ii) are independent the general solution is given by

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(i) and (ii) where c_1 and c_2 are arbitrary constants.

RULE III

Concider $\frac{dx}{P} = \frac{dy}{P} = \frac{dz}{R}$ — (i)

let P_1, Q_1, R_1 be functions od x, y and z then algebraically it can be shown that equation (i) =

$$\frac{P_1dx + Q_1dy + R_1dz}{P_1P + Q_1Q + R_1R} \quad \text{--- (ii)}$$

If $P_1P + Q_1Q + R_1R = 0$ in (ii) ~~then~~ then the numerator is also = 0 i.e.

$P_1dx + Q_1dy + R_1dz = 0$ which can be intergrated to give

$U_1(x, y, z) = c_1$

P_1, Q_1 and R_1 are called multipliers previous method could be used to obtain $U_2(x, y, z) = c_2$ or etter still get a new set f multiplies P_2, Q_2 and R_2.

Example 1

relevant multipliers to e

$$\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2}$$

oln

ing multipliers $x, -y, -z$

$$\frac{dx - ydy - zdz}{z(x+y) - yz(x-y) - z(x^2+y^2)} =$$

equation.

The denominator

$xz(x+y) - yz(x-y) - z(x^2+y^2) =$

$x^2z + xyz - xyz + y^2z - x^2z + zy =$

$xdx - ydx - zdz = 0$

$$\frac{x^2}{2} - \frac{y^2}{2} - \frac{z^2}{2} = \frac{c_1}{2}$$

$x^2 - y^2 - z^2 = c_1$ --- (i)

Again choosing multipliers

$y, x, -z$

$$\frac{ydx + xdy - zdz}{y^2(x+y) + xz(x-y) - z(x^2+y^2)}$$

then.

$ydx + xdy - zdz = 0$ integr

$$xy - \frac{z^2}{2} = \frac{c_2}{2}$$

$2xy - z^2 = c_2$ --- (ii)

Since (i) and (ii) are independ the complete solutions is given by (i) and (ii) c_1 and c_2 being arbitrary constant.

Example 2

Use multiplies to solve.

$$\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x}$$

Soln

Choosing multipliers 1, 1, 1

$$\frac{dx+dy+dz}{z-y+x-z+y-x}$$ = given equation.

peno = 0.

$dx+dy+dz=0$. Intergrating.

$x+y+z=c_1$ ---(i)

Again choosing multipler $x\ y\ z$

$$\frac{xdx+ydy+zdz}{xz-xy+xy-yz+yz-zx}.$$

$xdx+ydy+zdz=0$

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=c_2$$

$\Rightarrow\ x^2+y^2+z^2=c_2$ --(ii)

Since (i) and (ii) are independent the complete solution is given by i) and ii) c_1 and c_2 being arbitrary constant

RULE 4

from Rule 3 considering multiplier say P_2 Q_2 and R_2 which leads to

$$\frac{P_2dx+Q_2dy+R_2dz}{P_2P_1+Q_2Q_1+R_2R_1} \quad -(i)$$

where (i) is such that the numerator is an exact differential of the denominator, then (i) can be combined with one function from $\frac{dx}{P}+\frac{dy}{Q}+\frac{dz}{R}$ to obtain an Integral say $Q_2(x,y,z)=c_2$

Example

Solve

$$\frac{dx}{y^2(x-y)}=\frac{dy}{-x^2(x-y)}=\frac{dz}{z(x^2+y^2)}$$

Soln

Taking the first and the second fractions.

$$\frac{dx}{y^2}=\frac{dy}{-x^2}$$

$= x^2dx+y^2dy=0$ intergrate

$x^3+y^3=c_1$ ---(i)

Unlike rule (ii)

Choosing multipliers 1,-1,0

Each fraction in the given equation

$$\frac{dx-dy}{y^2(x-y)+x^2(x-y)}$$

$$\frac{dx-dy}{y^2x-y^3+x^3-x^2y}$$

$$\frac{dx-dy}{(x-y)(x^2+y^2)}$$

Combining this with the third fraction.

$$\frac{dx-dy}{(x-y)(x^2+y^2)}=\frac{dz}{z(x^2+y^2)}$$

$$\frac{dx-dy}{x-y} = \frac{dz}{z}$$. intergrating

$$\ln x-y = \ln z + \ln C_2$$

$$x-y = C_2 z \quad \text{---(ii)}$$

Since (i) and (ii) are independent. the Complete soln is given by ~~the~~ (i) (ii) where C_1 and C_2 are arbitrary constans.

TOTAL (PFAFFIAN) DIFFERENTIAL EQUATIONS

Definition: Let u_i, $i = 1,2,3 \ldots n$ be n functions of n independent Variables $x_1, x_2, x_3 \ldots x_n$ then Then

$$\sum_{i=1}^{n} u_i dx_i$$ is Called a

($\sum^{n} u\, dx_i = 0$ is a pfaffian differential equation)

Pfaffian differential equation

Considering three variables x, y, z the

$$Pdx + Qdy + Rdz = 0 \quad \text{--(i)}$$

nd P, Q and R are functions of x, y and z is Called a Pfaffian differential equation in ree variables x, y and z.

uation (i) can be intergreted irectly if there exists a unction U(x,y,z) whose total ifferential du is equal to dx + Qdy + Rdz

owever equation (1) may or may ot be integrable hence it is necessity to have an ntegrability Condition.

Necessary Condition for integrability

An equation of the form Pdx+Qdy+Rd is integrable if

$$P\left[\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right] + Q\left[\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right] + R\left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right]$$

✓✓

P	Q	R
x	y	z

$$P\left[\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right] + Q\left[\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right] + R\left[\frac{\partial P}{\partial y} - \frac{\partial Z}{\partial x}\right] = 0.$$

Example 1

$U = xy + xz$ then

$$du = ydx + xdy + zdx + xdz.$$

$$(y+z)dx + xdy + xdz = 0$$

for integrability

$$P\left[\frac{\partial Q}{\partial z} - \frac{\partial P}{\partial y}\right] + Q\left[\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right] + R\left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right] = $$

$$P = (x+z) \quad Q = x, \quad R = x$$

$$(y+z)[0-0] + x[1-1) + x[1-1) = 0.$$

$$0 + 0 + 0 = 0.$$

Hence $(y+z)dx + xdy + xdz = 0$ is integrable.

METHOD FOR SOLVIN TOTAL DE

1. Solution by Inspection

Sometimes by rearranging a given equation or by dividing through by function of x, y and we end up with exact differential [which are known to us)

The following are standard differentials

1. $$\frac{xdy - ydx}{x^2} = d\left[\frac{y}{x}\right]$$ recall, $$d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$$

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2. $\frac{xdy + ydx}{xy} = d\left[\ln \frac{y}{x}\right]$ $\begin{pmatrix} P & Q & R \\ x & y & z \end{pmatrix}$

3. $\frac{xdy - ydx}{x^2+y^2} = d\left[\tan^{-1}\left[\frac{y}{x}\right]\right]$ $\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right)$ $\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)$

4. $\frac{xdy + ydx}{xy} = d\left[\ln xy\right]$

5. $xdy + ydx = d[xy)$

6. $xydz + xzdy + yzdx = d[xyz)$

7. $2xdx + 2ydy + 2zdz = d[x^3+y^2+z^2]$

8. $\frac{f'(x,y,z)}{f(x,y,z)} = d\left[\ln f(x,y z)\right]$

$$\Rightarrow \int \frac{f'(x,y z)}{f(x,y,z)} = \ln(f(x,y,z)$$

Assignment.

Verify the integrability given

$zdx + xdy + 2(x+y+\sin z)\,dz = 0.$

$\begin{matrix} P & Q & R \\ x & y & z \end{matrix}$

$$P\left(\frac{\partial q}{\partial z} - \frac{\partial R}{\partial y}\right] + Q\left[\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right] + R\left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$P = z \quad Q = x \quad R = 2(x+y+\sin z).$

$$z\left(0 - \frac{\partial(2(x+y+\sin z)}{\partial y}\right] +$$

$$z\,\cancel{x}\left[\frac{\partial(2(x+y+\sin z)}{\partial x} - 1\right] + R\left[0 - 1\right]$$

$$z(-2) + x[2-1] + 2x + 2y + 2\sin z$$

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2. Solution by re-arranging and integr

Example.

Solve $(y^2+z^2-x^2)dx - 2\overset{x}{y}dy - 2xzdz = 0$

Solution

Check integrability.

$$P\left[\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left[\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) - R\left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right] = 0$$

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$P = y^2+z^2-x^2$

$Q = -2xy$

$R = -2xz.$

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$(y^2+z^2-x^2)[\cancel{Q} - 0] - 2xy[-2z \cancel{+} 2z] - 2xz[2y+2y$

~~$0 + 4xyz +$~~

0 ~~$-2xy$~~ $[-4z$

$0 + 2xy[-4z] - 2xz[4y]$

$0 + 8xyz - 8yz = 0.$

Hence it is integrable.

To Solve let us add and Subtract x^2dx

So that we can have.

$(y^2+z^2+x^2)dx - 2x^2dx - 2xydy - 2xzdz$

$(y^2+z^2+x^2)dx - 2x[xdx + ydy + zdz) = 0.$

$(x^2+z^2+x^2)dx = 2x[xdx + ydy + zdz)$

Re-arrange. ④

$$\frac{dx}{x} = \frac{2xdx + 2ydy + 2zdz}{x^2+y^2+z^2}.$$ ⑫

Intergrate.

$\ln x = \ln(x^2+y^2+z^2) + \ln C$ 6

$\Rightarrow x = C[x^2+y^2+z^2)$ which is a general Solution c by an aibiteraly constant.

xample 2.

Solve $yz^2[x^2 - yz)dx + zx^2[y^2 - xz]\overset{dy}{\wedge} + x^2y(z^2 - xy)\overset{dz}{\wedge} = 0$.

Solution

verify the integrability.

Solving

diving - through by $x^2y^2z^2$ we have.

$$\frac{[yx^2z^2 - y^2z^3]dx}{x^2y^2z^2} + \frac{[y^2x^2z - x^3z^2]dy}{x^2y^2z^2} + \frac{[xy^2z^2 - x^2y^3)dz}{x^2y^2z^2} = 0$$

$$\left[\frac{1}{y} - \frac{z}{x^2}\right]dx + \left[\frac{1}{z} - \frac{x}{y^2}\right]dy + \left[\frac{1}{x} - \frac{y}{z^2}\right]dz = 0$$

$$\frac{1}{y}dx - \frac{z}{x^2}dx + \frac{1}{z}dy - \frac{x}{y^2}dy + \frac{1}{x}dz - \frac{y}{z^2}dz = 0$$

$$\frac{1}{y}dx - \frac{x}{y^2}dy + \frac{1}{x}dz - \frac{z}{x^2}dx + \frac{1}{z}dy - \frac{y}{z^2}dz = 0$$

$$\frac{ydx - xdy}{y^2} + \frac{xdz - zdx}{x^2} + \frac{zdy - ydz}{z^2} = 0 \quad \checkmark$$

Integrating

$$\frac{x}{y} + \frac{z}{x} + \frac{y}{z} = c$$ which is the general Solution c being an arbiterary Constant.

Example 3

Find a function f(y) such that the total differential equation

$$\frac{yz + z}{x}dx - \cancel{z}dy + f(y)dz = 0$$

is integrable hence Solve it.

Solution

$$(yz + z)dx - xzdy + xf(y)dz = 0.$$

For intergrating

$$P\left[\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right] + Q\left[\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right]$$

$P = (yz + z) = \frac{\partial P}{\partial y} = z, \frac{\partial P}{\partial z} = (y+1)$

$Q = -xz \quad \frac{\partial Q}{\partial z} = -x, \frac{\partial Q}{\partial x} = -z$

$R = xf(y)$

$= \frac{\partial R}{\partial y} = xf'(y) = \frac{\partial R}{\partial x} = f(y)$

$= z(y+1)[-x - xf'(y)] - xz[f(y) - (y+1)] +$

$x[xf(y)[z + z) = 0$

$= -xyz - xyzf'(y) - xzf'(y) - xzf(y) + xyz + xz + 2xzf(y) = 0$

$\Rightarrow -xyzf'(y) - xzf'(y) + xzf(y) = 0$

$\Rightarrow -yf'(y) - f'(y) + f(y) = 0$

$= -f'(y)(y+1) + f(y) = 0$

$f(y) = f'(y)(y+1)$

but let $f'(y) = \frac{\partial f}{\partial y}$ and $f(y) = f$.

$f = (y+1)\frac{\partial f}{\partial y} \Rightarrow \frac{\partial y}{y+1} = \frac{\partial f}{f}$

integrating both Sides.

$\ln(y+1) = \ln f + \ln k$

$y + 1 = fk$ OR $f = k(y+1)$

$= f(y) = k(y+1)$

Our equation becomes.

$z(y+1)dx - xzdy + kx(y+1)dz = 0$

dividing by $xz(y+1)$

$$\frac{dx}{x} - \frac{dy}{y+1} + k\frac{dz}{z} = 0$$

integrating

$$\ln x - \ln(x+1) + k \ln z = \ln c$$

$$\ln x - \ln(y+1) + \ln z^k = \ln c$$

$\Rightarrow xz^k = c(y+1)$ which is the equal Solution kr c

HOMOGENEOUS EQUATION

The equation $Pdx + Qdy + Rdz = 0$ is homogeneous if, P, Q, R are homogeneous functions of the same degree in x, y and z

There are two methods for Solving homogenous equations.

Method 1

- Step 1 - Verify the ~~the~~ integrability

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right] - Q\left[\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right] + R\left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right]$$

Step 2 Calculate $Px + Qy + Rz$ and if it is not equal to zero then the Function $\frac{1}{Px + Qy + Rz}$ is taken as the integrating factor of the given equation.

We denote $Px + Qy + Rz$ as D So that $I.f = \frac{1}{D}$

Step 3
Mutiply the given equation by the integrating factor I.f $\left(\frac{1}{D}\right)$

Find $d(D)$ - total differential of D.

- Now add and substract the $d(D)$ from the Numerator. Write the given equation in the form.

$$\frac{d(D)}{D} \pm \ldots = 0 \quad \text{OR}$$

$$\frac{d(D) \pm \ldots}{D} = 0 \quad \text{then}$$

intergrate the result will most likely be an exact integral.

Example

Solve $(yz + z^2)dx - xzdy + xydz = 0$

Solution

Method 1

The equation is homogeneous and of degree 2.

Verify integrability.

$P = yz + z^2 \quad Q = -xz, \quad R = xy$

let $D = Px + Qy + Rz$

$$D = x(yz + z^2) - xzy + xyz$$

$$D = xz(y+z) + 0 + 0 \neq 0$$

Hence the $I.F = \frac{1}{xz(y+z)}$

we multiply the given equation by $\frac{1}{D}$

$$\frac{(yz + z^2)dx - xzdy + xydz}{D} \quad \ldots (i)$$

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Derivative of D

$$d(D) = d[xz(y+z)$$

$$= (y+z)\{zdx + xdz\} + xz\{dy+dz]$$

$$d(D) = z(y+z)dx + xzdy + (xy+2xz)dz$$

We now rewrite the given equation in the form.

$$\frac{d(D)}{D} - \frac{\left\{z(y+z)dx + xzdy + (xy+2xz)dz - z(y+z)dx + xzdy - xydz\right\}}{D} \quad \left[d(D) - \text{Equaton}\right]$$

$$\frac{d(D)}{D} - \frac{\{2xzdy + 2xzdz\}}{D} = 0$$

$$\frac{d(D)}{D} - \left[\frac{2xzdy + 2xzdz}{xz(y+z)}\right] = 0$$

Integrating

$$\ln D - 2\ln(y+z) = \ln C$$

~~by~~ but $D = xz(y+z)$

$$\ln xz(y+z) - \ln (y+z)^2 = \ln c$$

$$xz(y+z) = c(y+z)^2$$

$$xz = c(y+z)$$

hich is the general Solution c eing an arbitrary Constant.

METHOD 2

When $px + Qy + Rz = 0$ it is not possible to get an integrating factor.

We therefore use another method/techniques to solve the O.E.

STEP 1

verify integrability

STEP 2

let $x = uz \Rightarrow dx = zdu + udz$

$y = vz \Rightarrow dy = zdv + vdz$

and substitute in the given equation from which two Scenerios arise

Scenarion 1

If the coefficient of dz is zero then we remain with an equation in two variables u and v which can be solved by re-grouping.

Scenario 2

If the coefficient of dz is not zero we strive to separate z from u and v giving the result

$$\frac{f_1(u,v)du + f_2(u,v)dv}{f(u,v)} + \frac{dz}{z} = 0$$

Which can be solved directly by standard methods of Intergration.

Finaly replace u by $\frac{x}{z}$ and V by $\frac{y}{z}$ respectifully

Example

$(yz+z^2)dx - xzdy + xydz = 0$

Solution

$P = z(y+z)$ $Q = -xz$, $R = xy$.

verify the integrability.

Let $x = uz \Rightarrow dx = zdu + udz$

$y = vz \Rightarrow dy = zdv + vdz$

Substitute in the given equation

$(vz^2+z^2)\{zdu+udz\} - uz^2\{zdv+vdz\} + uvz^2dz = 0$

$\Rightarrow vz^3du + uvz^2dz + z^3du + uz^2dz - uz^3dv - uvz^2dz + uvz^2dz = 0$

$(vz^3+z^3)du - uz^3dv + (uvz^2+uz^2)dz = 0$

$z^3[v+1)du - uz^3dv + uz^2(v+1)dz = 0$

$$\frac{z^3(v+1)du}{uz^3(v+1)} - \frac{uz^3dv}{uz^3(v+1)} + \frac{uz^2(v+1)}{uz^3[v+1)}dz$$

$$\frac{du}{u} - \frac{dv}{v+1} + \frac{dz}{z} = 0$$

Integrating.

$\ln u - \ln(v+1) + \ln z = \ln c$

$uz = c(v+1)$

but $u = \frac{x}{z}$ and $v = \frac{y}{z}$

$\frac{x}{z}(z) = c\left(\frac{y}{z}+1\right)$

$x = c\left[\frac{y+z}{z}\right)$

$xz = c(y+z)$ which is the general Solution c being an arbitrary constant.

① $z^2dx + (z^2-2yz)dy + (2y^2-yz-xz)dz = 0$

② $2(y+z)dx - (x+z)dy + (2y-x+z)dz = 0$.

③ $yz(y+z)dx + zx(x+z) + xy(x+y)dz = 0$

④ $(y^2+yz)dx + (xz+z^2)dy + (y^2-xy)dz = 0$

REVISION

1. $$\frac{dx}{xz(z+xy)} = \frac{dy}{-yz(z^2+xy)} = \frac{dz}{x^4}$$

Soln

$x^4 - z$

$$\frac{dx}{xz(z+xy)} = \frac{dy}{-yz(z^2+xy)}$$

$$\frac{dx}{x^2} = \frac{dy}{-y}$$

$$\int\frac{dx}{x} + \int\frac{dy}{y} = 0.$$

$\ln x + \ln y = \ln c_1$

$xy = c_1$ --(i)

1 & 3

$$\frac{dx}{xz(z^2+c_1)} = \frac{dz}{x^4}$$

$$\frac{dx}{z(z^2+c_1')} = \frac{dz}{x^3}$$

$$\frac{x^4}{4} - \frac{z^4}{4} - \frac{z^2}{2}c_1 = \frac{c_2}{4}$$

$x^4 - z^4 - 2z^2xy = C_2$.

2. By choosing appropriate multipliers where necessary.

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$

Soln

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~~Lidx + Mi~~

Choose multipliers

$\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$.

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{1}{x}(x(y^2+z) - \frac{1}{y}(y(x^2+z)+\frac{1}{z}(z(x^2-y^2)} =$$

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0.$$

$$\ln|x| + \ln|y| + \ln|z| = \ln c_1$$

~~$x+y+z = c_1$~~

$$xyz = c_1$$

Second mutpliers.

$X, y, -1$

$$\int x\,dx + \int y\,dy - \int dz = 0$$

$$\frac{1}{2}x^2 + \frac{1}{2}y^2 - z = c_2$$

Pfaffian (Total) D.E

P	Q	R
x	y	z

$$P\left[\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right] - Q\left[\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right] +$$

$$R\left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right]$$

Example

1. Find f(y) such that the Pfaffian differential equation

$$(y^2+z^2-x^2)dx + 2xf(y)\,dy - 2xzdz = 0$$

$P = y^2+z^2-x^2$ $Q = 2xf(y)$ $R = -2xz$

~~$y^2+z^2-x^2$~~$[$

$\frac{\partial P}{\partial y} = 2y$ $\frac{\partial P}{\partial z} = 2z$

$\frac{\partial Q}{\partial x} = 2f(y)$ $\frac{\partial Q}{\partial z} = 0.$

$\frac{\partial R}{\partial x} = -2z$ $\frac{\partial R}{\partial y} = 0.$

$$(y^2+z^2-x^2)[0-0] + 2xf(y)[-2z-2z)$$

$$+ 2xz(2y - 2f(y)]$$

$$-8xzf(y) - 4xyz + 4\overset{x}{\wedge}z f(y) = 0$$

$$-4xzf(y) - 4xyz = 0.$$

$$f(y) + y = 0$$

$$\delta(y) = -y$$

Substitude where there is $\delta(y)$ by $-y$.

$$(y^2+z^2-x^2)dx - \overset{dy}{2xy} - 2xzdz = 0$$

$$(y^2+z^2+x^2)dx - 2x^2dx - 2xy\,dy -$$

$$2xzdz = 0.$$

$$(y^2+z^2+x^2)dx - 2x(xdx+ydy+zdz) = 0$$

$$(y^2+z^2+x^2)dx = 2x(xdx+ydy+zdz)$$

$$\frac{dx}{x} = \frac{2(xdx+ydy+zdz)}{(y^2+z^2+x^2)}$$

$$\int \frac{f'(x,y z)}{f(x y z)} = \ln f(x,y,z)$$

$$\ln x = \ln(x^2+y^2+z^2) + \ln c$$

$$x = c[x^2+y^2+z^2)$$

2. Solve by pfaffian

$$(z^2+xy)dx + (x^2+2yz)dy + (y^2+2xz)dz = 0.$$

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REVISION

Given two simultaneous equation $P_1 dx + Q_1 dy + R_1 dz = 0$ and $P_2 x + Q_2 dy + R_2 dZ = 0$. where P_1, Q_1, R_1 $P_2 Q_2$ and R_2 are function of x y ~~and~~ show that simultaneous pairs can be expressed in form

$$P_1 dx + Q_1 dy + R_1 dz = 0 \times Q_2$$

$$P_2 dx + Q_2 dy + R_2 dz = 0 \times Q_1$$

2. Solve the simutaneous equation

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x+y)^2}$$

Taking 1st & second

$$\frac{dx}{z} = \frac{dy}{-z}$$

$dx + dy = 0$ intergrating

$x + y = c_1$

Taking the 1st and 3rd

$$\frac{dx}{z} = \frac{dz}{(z^2 + c_1)^2}.$$

$$(x^2 + c_1)^2 x = z dz$$

Use of auxillary equation

Conceider $Pdx + Qdy + Rdz = 0$ (i) Its integrability condition is given by

$$P\left[\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right] + Q\left[\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right] + R\left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right] = 0 \quad \text{(ii)}$$

Campairing (i) and (ii) we obtain a set of simutaneous equation known as the auxiliary equation given by:

$dz = 0$

$z = c$ $(z = 0$

$$\frac{\partial x}{\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}} = \frac{\partial y}{\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}} = \frac{\partial z}{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}} \quad dz =$$

which can be solved as previous discussed to obtain $f(x, y, z) = C_1$ and $f(x, y, z) = C_2$.

let $U = c_1$ and $V = c_2$ be two integrals so obtained with the $Adu + Bdv = 0$ - (iv)

We compaire (i) and (ii) to obtain A and B put the value of A and B in (iv) then integrale

Now substitute the value of A and B in the relation after intergrating giving the required general solution.

NOTE: This method come in handy where methods discussed before cannot be used.

Example

Solve ~~to~~ $xz^3 dx - zdy + 2ydz = 0$

Solution:

Example

Solve $xz^3dx - zdy + 2ydz = 0$

Solution

Compare with $Pdx + qdy + Rdz = 0$

$P = xz^3 \quad q = z \quad R = 2y$

$\frac{\partial P}{\partial z} = 3xz^2, \quad \frac{\partial P}{\partial y} = 0$

$\frac{\partial q}{\partial z} = -1 \qquad \frac{\partial q}{\partial x} = 0$

$\frac{\partial R}{\partial y} = 2 \qquad \frac{\partial R}{\partial x} = 0$

$$\frac{dx}{\frac{\partial q}{\partial z} - \frac{\partial R}{\partial y}} = \frac{dy}{\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}} = \frac{dz}{\frac{\partial P}{\partial y} - \frac{\partial q}{\partial x}}$$

Now

P q R

$$\frac{dx}{-1-2} = \frac{dy}{0-3xz^2} = \frac{dz}{0-0}.$$

$$\frac{dx}{-3} = \frac{dy}{-3xz^2} = \frac{dz}{0}$$

From the 3rd relation.

$dz = 0; \quad z = c_1$

$u = z = c_1$

du = dz

Taking 1st and and 2nd

$\frac{dx}{1} = \frac{dy}{c_1^2 x} =$ Since $z = c_1$

$c_1^2 x dx = dy$ Integrating

$\frac{c_1^2 x^2}{2} = y + c_2 \Rightarrow c_2 = \frac{1}{2}c_1^2x^2 - y$

$\Rightarrow c_2 = c_1^2x^2 - 2y$

But $c_1 = z \Rightarrow c_2 = z^2x^2 - 2y$

Let $V = c_2 \Rightarrow V \Rightarrow z^2x^2 - 2y$

$dV = 2zx^2dz + 2xz^2dx - 2dy$

We formulate $Adu + Bdv = 0$.

$Adz + B[2x^2zdz + 2xz^2dx - 2dy] = 0$

$2Bxz^2dx - 2Bdy + [A + 2Bx^2z]dz = 0$

Compairing the given equation
$xz^3dx - zdy + 2ydz = 0$

$2Bxz^2 = xz^3 \Rightarrow 2B = z \Rightarrow B = \frac{1}{2}z$

But $z = u \Rightarrow B = \frac{1}{2}u$

Also. $A + 2Bx^2z = 2y$

$A + 2\left[\frac{z}{2}\right)x^2z = 2y$

$A + x^2z^2 = 2y \Rightarrow A = 2y - x^2z^2$

$\Rightarrow A = -V$

But $Adu + Bdv = 0$

$\Rightarrow -Vdu + \frac{1}{2}udv = 0.$

$udv = 2Vdu \Rightarrow \frac{dv}{v} = 2\frac{du}{u}$

integrating

$\ln v = 2\ln u + \ln k$

$V = ku^2$

But we know that $V = z^2x^2 - 2y$
and $u = z$

$z^2x^2 - 2y = kz^2$ which is the general Solution k being an arbitrary Constant.

General method

$Pdx + qdy + Rdz = 0.$

⇒ In this, method we take one Variable to be a Constant.

Step 1: Verify integrability.

Step 2: Treat one of the variable Say z as a Constant so that $dz = 0$, leaving us with
$Pdx + qdy = 0$ ---(1)

+ we need to carefuly Select the variable to be made

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a constant so that the remaining terms as easily integrable.

Step 3: Let the solution of equation (1) be a function of x and y, i.e a function $f(x,y) = f(z)$ where $f(z)$ is an arbitrary fuction of z.

$$u(x,y) = f(z) \quad ---(2)$$

Step 4: We now differentiate equation (2) totally w.r.t x, y and z and compare the results with $Pdx + Qdy + Rdz = 0$.

After this comparison we shall have an equation in two variation f and z.

. If the coefficient of df and dz involve x and y it will always be possible to releave them using 2.

Step 5 $Adu + Bdv = 0$

Solve the equation in Step 4 and obtain the value of f and put the value in (2) to get the general solution.

Example $c_1 = u$ $c_2 = (v)$

Solve $3x^2dx - 3y^2dy - [x^3+y^3+e^{2z}]dz = 0$

Soln

1st verify integrability

Compare with $Pdx + Qdy + Rdz = 0$.

$P = 3x^2$, $R = -(x^3+y^3+e^{2z})$

$Q = -3y^2$

let z be a constant $dz = 0$, then given equation becomes.

$$3x^2dx + 3y^2dy = 0 \quad ---(2)$$

Integrability

$$x^3 + y^3 = f(z) \quad --(3)$$

Where the constant of integrability is $f(z)$ since we treated z as a constant

- Differentiate (3) w.r.t $x, y,$ and z

$$3x^2dx + 3y^2dy - f'(z)\,dz = 0 \quad --4$$

Compare (4) and (1)

$f'(z) = x^3 + y^3 + e^{2z}$ since $f(z) = x^3+y^3$

$f'(z) = f(z) + e^{2z}$ since $f(z) = x^3+y^3$

$\Rightarrow \frac{df}{dz} = f + e^{2z}$ which is a linear ODE in terms of f and z

$\frac{df}{dz} - P(z)f = Q(z)$

$\frac{df}{dz} = -f \neq e^{2z}$

Compare $\frac{df}{dz} - P(z)f = Q(z)$

$\frac{df}{dz} - f = e^{2z}$

$I.F = e^{\int P(z)dz} = e^{\int(-1)dz} = e^{-z}$ the.

Solution $(I.F)f = \int (I.F)Q(z)$

Solution $\Rightarrow If = \int (I.f)\,Q(z)$

$\Rightarrow e^{-z}f = \int e^{-z}(e^{2z})dz = \int e^{z}dz$

$fe^{-z} = e^{z} + c$ $e^{\int P(z)dz}$, e^{-1}

$\Rightarrow f = e^{2z} + c^{z}$

$(I.F)f = \int (I.F)Q(z)$

Hence the general solution is given by (3),

$x^3 + y^3 = e^{2z} + Ce^{z}$; C being an arbitrary constant

Quiz

1) $(2x + y^2 + 2xz)dx + 2xy\,dy + x^2dz = 0$

2) $(2x^2 + 2xy + 2xz^2 + 1)dx + dy + 2zdz = 0$

✓ 3) $(x^2 + y^2 + z^2)dx - 2xydy - 2zdz = 0$

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Q2

By taking one variable to a constant

$(2x+y^2+2xz)dx+2ydy+x^2dz=0$

Solve the total differential equation.

1). verify the integrability.

let x be a constant $\Rightarrow dx=0$

$2ydy+x^2dz=0$ ... (i)

$y^2+x^2z=f(x)$

$f(x)=y^2+x^2z$ ... (ii)

differentiate (ii) w.r.t x y and z

$f'(x)=2y\,dy+x^2dz$ ... (iii)

$-f'(x)dz+2ydy+x^2dz=0$ ... (iv)

4) Compaire (i) and (iv)

$-f'(x)=2x+y^2+2xz$

$-f(x)=2x+y^2+2xz$

P Q R

x y z

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left[\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)$$

$$+R\left(\frac{\partial P}{\partial y}-\frac{\partial R}{\partial x}\right)$$

Question 3

a) Solve the Pfaffian differential equation:

$(y.z+xyz)dx+(zx+xyz)dy+(xy+xyz)dz=0$

b) Find $f(y)$ such that the Pfaffian differential equation

$\{(yz+z)/x\}dx-zdy+f(y)dz=0$

is integrable hence solve it.

c) Use Langrange's method to solve $xyp+y^2q=2xy-2x^2$.

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ORTHOGONAL TRAJECTORIES OF A SYSTEM OF CURVES ON A SURFACE

let the given surface be a function $f(x,y,z) = 0$ (i) and let the system of curves lying on the surface (i) be the curves of intersection of (1) and another function $\phi(x,y,z) = c$, c-parameter. & The direction ratios dx, dy and dz of the tangents at any point (x,y,z) on the given curve lying on the surface (i) and (ii) are given by

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0 \text{ and}$$

$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0$$

Solving for dx, dy and dz we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad ---(3)$$

P Q R
x y z

where

$$\left.\begin{aligned} P &= \frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial z} - \frac{\partial f}{\partial z}\cdot\frac{\partial \phi}{\partial y} \\ Q &= \frac{\partial f}{\partial z}\cdot\frac{\partial \phi}{\partial x} - \frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial z} \\ R &= \frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y} - \frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial x} \end{aligned}\right] - ④$$

Thus P, Q and R are directional ratios of the tangents.
By definition P, Q, R will be direction ratios of the normal of the required orthogonal trajectories whose differential equations are given by

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0 \text{ and}$$

$$Pdx + Qdy + Rdz = 0.$$

Upon solving for dx, dy, dz we obtain another system

$$\frac{dx}{P'} = \frac{dy}{Q'} = \frac{dz}{R'} \quad ... ⑤$$

$$P' = R\frac{\partial f}{\partial y} - Q\frac{\partial f}{\partial z},\quad Q' = P\frac{\partial f}{\partial z} - R\frac{\partial f}{\partial x}$$

$$R' = Q\frac{\partial f}{\partial x} - P\frac{\partial f}{\partial y}$$

The solution to (i) and the surface (i) gives the required orthogonal trajectory.

Example

Find the orthogonal trajectories on the cone $x^2+y^2 = z^2\tan^2\alpha$ of its intersection with the family of planes parallel to $z=0$

Solution

The surface $f(x,y,z) = 0$ is given by
$x^2+y^2-z^2\tan^2\alpha = 0$ . . . (i)

The family of planes parallel to $z=0$ is $z = c$, c - is a parameter ..(2)

The system of the ~~intersection~~ differential equation of the curves 1 and 2 are

$$2xdx + 2ydy + 2z\tan^2\alpha\, dz = 0 \text{ and}$$

$$0dx + 0dy + dz = 0.$$

Since
$dz = 0 \Rightarrow 2xdx + 2ydy = 0.$
$\Rightarrow 2xdx = -2ydy$ giving us a

$$\frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{0}$$

The system of differential equations of the required orthogonal trajectories is given by

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0 \text{ and}$$

$$Pdx + Qdy + Rdz = 0.$$

hence

$xdx+ydy-z\tan^2\alpha dz=0$ ✓ and

$ydx-xdy+0dz=0$ ✓

Solving for dx, dy, dz

$$\frac{dx}{xz\tan^2\alpha}=\frac{dy}{yz\tan^2\alpha}=\frac{dz}{x^2+y^2}$$

NOTE

$xdx+ydy-z\tan^2\alpha dz=0 \cdot \times y$

$ydx+xdy+0dz=0 \quad \times x$

$xydx+y^2dy-yz\tan^2\alpha dz=0$

$xydx+x^2dy-0dz=0$ −

$(x^2+y^2)dy-yz\tan^2\alpha dz=0$

$(x^2+y^2)dy=yz\tan^2\alpha dz$

~~Also~~ $\frac{dy}{yz\tan^2\alpha}=\frac{dz}{x^2+y^2}$ - - - ③.

Also ✓ ✓

$x^2dx+xydy-xz\tan^2\alpha dz=0$.

$y^2dx-xydy-0dz=0$ +

$(x^2+y^2)dx-xz\tan^2\alpha dz=0$.

$\frac{dx}{xz\tan^2\alpha}=\frac{dz}{x^2+y^2}$ - - ④

From ③ and ④

$$\frac{dx}{xz\tan^2\alpha}=\frac{dy}{yz\tan^2\alpha}=\frac{dz}{x^2+y^2}$$

Solving

Taking 1st and 2nd

$\frac{dx}{x}=\frac{dy}{y}$ integrating

$\Rightarrow \ln x=\ln y+\ln c_1 \Rightarrow x=c_1y$ or

$\frac{x}{y}=c_1$

Taking multipliers $x, y, 0$

Then $\frac{xdx+ydy}{(x^2+y^2)z\tan^2\alpha}=\frac{dz}{x^2+y^2}$

Since multipliers P_1 Q_1 R_1 give.

$$\frac{P_1dx+Q_1dy+R_1dz}{P_1P+Q_1Q+R_1R}$$

$xdx+ydy=z\tan^2\alpha dz$ intergrating

$\frac{x^2}{2}+\frac{y^2}{2}=\frac{z^2}{2}\tan\alpha$ hence

$x^2+y^2=z^2\tan^2\alpha$

Hence the required surfaces are

$\frac{x}{y}=c$ and $x^2+y^2=z^2\tan^2\alpha$.

Example 2

Find the orthogonal trajectory of a Concoid $(x+y)z=1$ of the Conics in which it is cut by a system $x-y+z=k$ where k is a parameter.

PARTIAL DIFFERENTIAL EQUATIONS

A P.D.E is linear if the dependent variable and its derivatives only occurs in the first degree and are not multiplied togethe. otherwise the P.D.E is non-linear.

Notations

Lets Consider three variables x y and z, z being dendent and x, and y being independent i.e $z = f(x,y)$

The following standard notation are used

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r$$

$$\frac{\partial^2 z}{\partial x \partial y} = s \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} = t$$

$$pq + rt + sq = 0$$

Also

$$\frac{\partial u}{\partial x} = u_x \quad \frac{\partial V}{\partial x} = V_x, \quad \frac{\partial^2 u}{\partial x^2} = u_{xx}$$

$$\frac{\partial^2 V}{\partial x^2} = V_{xx} \quad \frac{\partial^2 u}{\partial x \partial y} = u_{xy}$$

Recall $P[q_z - R_y] + Q[R_x - P_z] + R[P_y - Q_x]$.

CLASSIFICATIONS OF 1ST ORDER PDES

CLASSIFICATION OF FIRST ORDER PDE

Linear equation

- A first order equation of the form $f(x,y,z,P,Q) = 0$ is linear if it is linear in P, q and z e.g.

$$yx^2p + xy^2q = xyz + x^2$$

$$\Rightarrow P(x,y)p + Q(x,y)q = R(x,y)z + S(x,y)$$

Semi-linear

$f(x,y,z,P,q) = 0$ is semi linear if it is linear in P and q and the coefficient of P and Q are function of x and y only

$$\Rightarrow P(x,y)p + Q(x,y)q = R(x,y,z)$$

e.g

$$xyp + x^2yq = x^2y^2z^2$$

Quasi linear

- A Function $f(x,y,z,P,Q)$ is Quasi linear if it is linear in p and q only.

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$$

$$x^2yzp + xyz^2q = xyz$$

Non-linear

Any equation that does not fall in the categories above is said to be non-linear

i.e not linear in p and q

$$p^2 + q^2 = 1$$

(1

ORIGIN OF PDES

DERIVATION OF PDES

CASE 1

Arbitrary constants < independent Variable

e.g

$$z = ax + y$$

Solution

Differentiating w.r.t x $\Rightarrow \frac{\partial z}{\partial x} = a$ -(1)

Differentiating w.r.t y $= \frac{\partial z}{\partial y} = 1$ --(2)

using eqn (1) in the given equation

$$z = x\frac{\partial z}{\partial x} + y \quad \text{---(3)}$$

Hence, the PDEs formed are (2) (3)

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N/B This category yields more than one PDE of order 1

- We form PDEs by eliminating the arbitrary constants.

CASE II

Arbitrary Constants = Independent variables
e.g for a PDE by eliminating the arbitrary constant given.

$az + b = a^2x + y$

Soln

Differentiating w.r.t x ay y

$a\frac{\partial z}{\partial x} = a^2 \Rightarrow \frac{\partial z}{\partial x} = a$ -(i)

$a\frac{\partial z}{\partial y} = 1$ - - 2

Eliminating a from (1) and (2)

From (2) $a = \frac{1}{\frac{\partial z}{\partial y}}$ substituting in (1)

$$\frac{\partial z}{\partial x} = \frac{1}{\frac{\partial z}{\partial y}}$$

$$\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) = 1 \text{ or } pq = 1$$

which is a unique P.D.E.

N/B: This Category yields a unique partial differential equation.

Example 2

$z = ax + by$

Differentiating w.r.t x and y.

$\frac{\partial z}{\partial x} = a$ -(i) and $\frac{\partial z}{\partial y} = b$ --(2)

Substituting in the given equation

$z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$ which is a unique differential equation.

CASE III

Arbitrary Constant > Independent Variables

- Example

Eliminating a, b and c from

$z = ax + by + cxy$

Soln

Differentiating w.r.t x and y more than one.

$\frac{\partial z}{\partial x} = a + cy$ -(1) a – ②

c ≡ ↗ (a) (b)

$\frac{\partial z}{\partial y} = b + cx$ --(2)

$\frac{\partial^2 z}{\partial x^2} = 0$ --(3)

$\frac{\partial^2 z}{\partial y^2} = 0$ (4)

$\frac{\partial^2 z}{\partial x \partial y} = c$ -(5)

We need to obtain the values of a and b using given equations.

– Using (1) and (5)

$$a = \frac{\partial z}{\partial x} - y\frac{\partial^2 z}{\partial x \partial y}, \quad b = \frac{\partial z}{\partial y} - x\frac{\partial^2 z}{\partial x \partial y}$$

Substituting in the given equation.

$$z = x\left[\frac{\partial z}{\partial x} - y\frac{\partial^2 z}{\partial x \partial y}\right] + y\left[\frac{\partial z}{\partial y} - x\frac{\partial^2 z}{\partial x \partial y}\right] + xy\frac{\partial^2 z}{\partial x \partial y}$$

$$z = x\frac{\partial z}{\partial x} - xy\frac{\partial^2 z}{\partial x \partial y} + y\frac{\partial z}{\partial y} - xy\frac{\partial^2 z}{\partial x \partial y} + xy\frac{\partial^2 z}{\partial x \partial y}$$

$$z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - xy\frac{\partial^2 z}{\partial x \partial y} \quad \text{- - - (i)}$$

Hence we have 3 PDEs given by (3),(4) and (6)

N/B The Category yields equations of order greater than one.

DERIVATION OF PDEs, BY ELIMINATING ARBITRARY FUNCTIONS $\phi(u,v)=0$ WHERE U AND V ARE FUNCTIONS OF x, y and z

Example

Form a PDE by eliminating an arbitrary Function ϕ from
$\phi(x+y+z, x^2+y^2-z^2)=0$

State the order of the DE

Solution

$$\frac{\partial\phi}{\partial u}\left[\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right]+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)$$

$$\frac{\partial\phi}{\partial u}\left[\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right]+\frac{\partial\phi}{\partial v}\left[\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right]$$

Given $\phi(x+y+z, x^2+y^2-z^2)=0$ ---(1)

Compare to $\phi(u,v)=0$ ---(2)

$u=x+y+z$ and $V=x^2+y^2-z^2$.(3).

Differentiating (2) w.r.t. x we get.

$$\frac{\partial\phi}{\partial u}\left[\frac{\partial u}{\partial x}+\frac{p\partial u}{\partial z}\right]+\frac{\partial\phi}{\partial v}\left[\frac{\partial v}{\partial x}+\frac{p\partial v}{\partial z}\right]=0$$

$$\frac{\partial u}{\partial x}=1,\quad \frac{\partial u}{\partial z}=1\quad \frac{\partial v}{\partial x}=2x,\quad \frac{\partial v}{\partial z}=-2z$$

$$\frac{\partial\phi}{\partial u}[1+p]+\frac{\partial\phi}{\partial v}[2x-2zp]=0.$$

$$\frac{\partial\phi}{\partial u}[1+p]=\frac{\partial\phi}{\partial v}[2zp-2x]$$

$$\Rightarrow \frac{\partial\phi}{\partial u}\Big/\frac{\partial\phi}{\partial v}=\frac{2zp-2x}{1+p}$$

Differentiating (2) w.r.t y

$$\frac{\partial\phi}{\partial u}\left[\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right]+\frac{\partial\phi}{\partial v}\left[\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right]$$

$$\frac{\partial u}{\partial y}=1,\ \frac{\partial u}{\partial z}=1,\ \frac{\partial v}{\partial y}=2y\quad \frac{\partial v}{\partial z}=-2z$$

$$\frac{\partial\phi}{\partial u}[1+q]+\frac{\partial\phi}{\partial v}[2y-2zq]=0$$

$$\Rightarrow \frac{\partial\phi}{\partial u}\Big/\frac{\partial\phi}{\partial v}=\frac{2zq-2y}{1+q}$$

Compairing (4) and (5)

$$\frac{2zp-2x}{1+p}=\frac{2zq-2y}{1+q}$$

$$(1+q)(zp-x)=(1+p)(zq-y)$$

Which is the required differential equation.

Alternative.

If $\phi(u,v)=0$ then $u=f(v)$ and $V=f(u)$ where u and v are functions of x, y and z.

-Given $\phi(x+y+z, x^2+y^2-z^2)=0$

$$\Rightarrow x+y+z=f(x^2+y^2-z^2)\ \text{--}(1)$$

Differentiating (1) w.r.t x $\quad \frac{\partial}{\partial x}(zz)=2x\ \ z\frac{\partial z}{\partial x}+z\frac{\partial z}{\partial x}$

$$(1+p)=f'(x^2+y^2-z^2)(2x-2zp)$$

Differentiating w.r.t y

$$1+q=f'(x^2+y^2-z^2)(2y-2zq)\ \text{----}(3)$$

From (2) $f'(x^2+y^2-z^2)=\dfrac{1+p}{2x-2zp}$.

From (3) $f'(x^2+y^2-z^2)=\dfrac{1+P}{2y-2zq}$

Compairing

$$\frac{1+p}{2x-2zp}=\frac{1+q}{2y-2zq}$$

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$(1+p)(2y-2zq)=(1+q)(2x-2zp)$

LINEAR PARTIAL DIFFERENTIAL EQUATION

A Quasi-linear P.D.E of order 1 is of the form $Pp+Qq=R$ where P, Q, R are function of x, y and z and is known as the Langrange equation

LAGRANGE'S EQUATION

The general Solution of the Langranges equation $Pp+Qq=R$ is given by $\phi(u,v)=0$ where ϕ is an arbitrary function of and $U(x,y,z)=C_1$ and $V(x,y,z)=C_2$ which are two independent Solution of $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ and C_1 and C_2 are arbitrary Constants and at least one of u and v must contain z.

Solution steps:

Step 1

Re-write the given P.D.E in the form $Pp+Qq=R$ [standard form) - (i)

Step 2

Write down the Langranges auxillary equation

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \quad --(2)$$

Step 3

Solve (2) using known method to obtain $U(x,y,z)=C_1$ and $V(x,y,z)=C_2$ being two independent solutions.

The integral of (1) is therefore in the Form $\phi(u,v)=0$ or $f(u)=v$ or $u=f(v)$, ϕ and f being arbitrary Constants.

Example

- Solve the langrange's equation $\frac{y^2z}{x}p+xzq=y^2$

Solution

- we re-write.

$y^2zp+x^2zq=xy^2$ compare $Pp+Qq=R$

- $P=y^2z$, $Q=x^2z$, $R=xy^2$

The langranges equation is

$$\frac{dx}{y^2z}=\frac{dy}{x^2z}=\frac{dz}{xy^2}$$

Taking the first and second.

- $\frac{dx}{y^2}=\frac{dy}{x^2} \Rightarrow x^2dx-y^2dy=0$

interegrating

$x^3-y^3=C_1 \quad ---(i)$

- Again taking 1st and 3rd Fraction

$\frac{dx}{z}=\frac{dz}{x} \quad xdx+zdz=0$

interegrating

$x^2-z^2=C_2 \quad ---(ii)$

$\therefore$ The required integral is in the form $\phi(u,v)=0$

- $\phi(x^3-y^3, x^2-z^2)=0$ OR

$x^3-y^3=f(x^2-z^2)$

→ ϕ being an arbitrary Const

- → f – '' '' ''

Confirming

✓✓

$$\frac{\partial\phi}{\partial u}\left[\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right]+\frac{\partial\phi}{\partial v}\left[\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right]=0$$

$u=x^2-y^3 \quad \frac{\partial u}{\partial x}=3x^2, \frac{\partial u}{\partial z}=0$

$V=x^2-z^2, \frac{\partial v}{\partial x}=2x, \frac{\partial v}{\partial z}=-2z$

$$\frac{\partial\phi}{\partial u}[3x^2+0)+\frac{\partial\phi}{\partial v}(2x-2zp)=0$$

$$\frac{\partial\phi}{\partial u} \Big/ \frac{\partial v}{\partial v} = \frac{2zp - 2x}{3x^2}.$$

w.r.t y.

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right] + \frac{\partial\phi}{\partial u}\left[\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right] = 0.$$

$$\frac{\partial u}{\partial y} = -3y^2;\ \frac{\partial y}{\partial z} = 0\ \ \frac{\partial v}{\partial y} = 0\ \frac{\partial v}{\partial z} = -2z$$

$$\frac{\partial\phi}{\partial u}\left[-3y^2 + 0\right] + \frac{\partial\phi}{\partial v}\left[0 - 2zq\right] = 0$$

$$\frac{\partial\phi}{\partial u} \Big/ \frac{\partial\phi}{\partial v} = \frac{-2zq}{3y^2}$$ Comparing the two equation.

$$\frac{-zq}{y^2} = \frac{zp - x}{x^2} \Rightarrow -x^2 zq = y^2 zp - xy^2$$

$$= y^2 zp + x^2 zq = xy^2$$

Alternative 2

$$x^2 - z^2 = f(x^3 - y3)$$

$$2x - 2zp = f'(x^3 - y^3)[3x^2]$$

$$f'(x^3 - y3) = \frac{2(x - zp)}{3x^2}.$$

$$0 - 2zq = f'(x^3 - y^3)[-3y^2)$$

$$f'(x3 - y3) = \frac{2zq}{3y^2} \quad \cdots ②$$

$$\frac{x - zp}{x^2} = \frac{zq}{y^2}$$

$$= y^2 zp + x^2 zq = xy^2$$

Example 2

Solve $y^2p - xyq = x(z - 2y]$.

Solution

Compare $Pp + Qq = R$

$$P = y^2 \quad Q = -xy \quad R = x(z - 2y]$$

The Langrange's auxilliary equation.

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z - 2y)}$$

Taking 1st two

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

$$x\,dx + y\,dy = 0 \Rightarrow \text{integrating}$$

$x^2 + y^2 = c_1$ c_1 - Arbitrary Constant.

Taking 3rd and 2nd.

$$\frac{dy}{-y} = \frac{dz}{z - 2y} \qquad Pp + Qq = R$$

Its not separable hence

$$\frac{dz}{dy} = \frac{z - 2y}{-y} = -\frac{z}{y} + 2$$

$$\frac{dz}{dy} + \frac{1}{y}(z) = 2$$ which is linear equation in z and y.

linear eqn

~~Not~~ $\alpha | \beta$

linear equation are in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$\frac{dz}{dy} + P(y)z + Q(y)$$

$$If\,(y) = \int (If)(Q)\,dx + c$$

$$I.f = e^{\int px} \qquad \frac{dz}{dy} + g(y)z = f(y)$$

$$\frac{dz}{dy} + \frac{1}{y}[z] = 2 \qquad g(y) = \frac{1}{y}\ \ f(y) = 2$$

$$If = e^{\int g(y)dy}$$

$$I.F = e^{\int \frac{1}{y}dy} = e^{\ln y} = y$$

Solution is $(I.F)z = \int f(y)(I.f)\,dy + c$

$$yz = \int 2y\,dy = yz = y^2 + c_2$$

$yz - y^2 = c_2$ c_2 being arbitrary Constant.

What is a separable equation

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NON-LINEAR PDE'S

COMPATIBLE SYSTEM of 1st ORDER EQUATIONS

Consider 1st Order PDE's

$f(x,y,z,p,q)=0 \quad - - ①$

$g(x,y,z,p,q)=0 \quad - - ②$

Equations ① and ② are known to be compatible if every solution of ① is also a solution of ②

To solve such equations we use the Jacobian matrix of f and g given by

$$\frac{\partial(f,g)}{\partial(p,q)} \neq 0$$

Upon simplification we expect the jacobian determinants

$$[fg] = \frac{\partial(fg)}{\partial(x\underline{p})} + p\frac{\partial(f,g)}{\partial(\underline{z},p)} + \frac{\partial(fg)}{\partial(\underline{y}q)} + q\frac{\partial(fg)}{\partial(z,q)} = 0$$

Example

Show that the equations $xp = yq$ and $z(xp+yq) = 2xy$ are compatible hence solve

Solution

$f(x,y,z,p,q) = xp - yq = 0 \quad - - (i).$

$g(x,y,z,p,q) = z(xp+yq) - 2xy = 0 \quad -②$

For Compatibility $(f,g) = 0$.

$$[fg] = \frac{\partial(fg)}{\partial(xp)} + p\frac{\partial(fg)}{\partial(z,p)} + \frac{\partial(fp)}{\partial(yq)} + q\frac{\partial(fg)}{\partial(z,q)} = 0.$$

$$\frac{\partial[fg)}{\partial(x,p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} p & x \\ zp-2y & xz \end{vmatrix}$$

$$= xzp - x[-zp-2y] = 2xy$$

$$\frac{\partial(fg)}{\partial(zp)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} 0 & x \\ xp+yq & xz \end{vmatrix}$$

✓

$x[zp - zp + 2y$

$$= -x^2p - xyq$$

$$\frac{\partial(fg)}{\partial(yq)} = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} -q & -y \\ zq-2x & yz \end{vmatrix}$$

✓ $= -yzq + yzq - 2xy = -2xy$

$$\frac{\partial(fg)}{\partial(z,q)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & -y \\ xp+yq & yz \end{vmatrix}$$

$$= xyp + y^2q$$

$[fg] = 2xy + p[-x^2p - xyq) - 2xy + q(xyp + y^2q)$

$= -x^2p^2 - xypz = (yp+xy)(yq-x$

but

$xp = yq \Rightarrow yq - xp = 0.$

$\Rightarrow (yq+xp)(0) = 0.$ ✓

We now solve p and q in (1) and 2

$xp - yq = 0$ ✓

$+$

$xp + yq = \frac{2xy}{z}$ ✓

$xp = \frac{2xy}{z} \Rightarrow p = y/z$

From (1)

$yq = xp \Rightarrow q = \frac{xp}{y}$

but $p = y/z \Rightarrow q = \frac{xy}{z} \times 1/y = \frac{x}{z}$

We now set up the

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$$dz = p\,dx + q\,dy$$

$$\Rightarrow dz = \frac{y}{z}dx + \frac{x}{z}dy$$

$z\,dz = y\,dx + x\,dy$. intergrating

$$\frac{z^2}{2} = xy + C$$

$\Rightarrow z^2 = 2xy + C$, C arbitrary ✓✓
Constant.

N/B
Largely use Crammers rule.

$$\begin{matrix} 2x+y = 5 \\ x+4y = 6 \end{matrix} \Bigg\} \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

$$X = \frac{\begin{vmatrix} 5 & 1 \\ 6 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix}} = \frac{14}{7} = 2$$

$$Y = \frac{\begin{vmatrix} 2 & 5 \\ 1 & 6 \end{vmatrix}}{7}\ \frac{7}{7} = 1$$

Now use Crammers

$$\begin{bmatrix} x & -y \\ x & y \end{bmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{2xy}{z} \end{pmatrix} ✓$$

$$X = \frac{\begin{vmatrix} 0 & -y \\ \frac{2xy}{z} & y \end{vmatrix}}{\begin{vmatrix} x & -y \\ x & y \end{vmatrix}} = \frac{\frac{2xy^2}{z}}{2xy} = \frac{y}{z}$$

$$y = \frac{\begin{vmatrix} x & 0 \\ x & \frac{2xy}{z} \end{vmatrix}}{2xy} = \frac{2x^2y}{z} \times \frac{1}{2xy} = \frac{x}{z}$$

$$X = \frac{\begin{vmatrix} 0 & -y \\ \frac{2xy}{z} & y \end{vmatrix}}{\begin{vmatrix} x & -y \\ x & y \end{vmatrix}}$$

$$Y = \frac{\begin{vmatrix} x & 0 \\ x & \frac{2xy}{z} \end{vmatrix}}{\begin{vmatrix} x & -y \\ x & y \end{vmatrix}}$$

Langranges equation

Exercise

1. $p\tan x + q\tan y = \tan z$
2. $x^2p + y^2q = z^2$
3. $p + 3q = 5z + \tan(y - 3x]$
4. $xyp + y^2q = zxy - 2x^2$
5. $xp - yq = xy$
6. $p - q = \frac{z}{x+y}$

Use multiplers

7) $z(x+y)p + z(x-y)q = x^2+y^2$
8) $x(y^2-z^2)q - y(z^2+x^2)q = z(x^2+y^2)$
9) $(y-zx)p + (x+yz)q = x^2+y^2$

Exercise 2

✓ Show that the equations are compatible then solve.

1) $f = xp - yq = x$ and $g = x^2p + q = xz$
2) $p^2 + q^2 = 1$ and $(p^2+q^2)x = pz$
3) $z = px + qy$. and $2xy(p^2+q^2) = z(yp+xq$
4) $p = x^4 - 2xy^2 + y^2$ and
$g = 4xy^3 - 2x^2y - \sin y$

Langranges equation Solutions

1. $p\tan x + q\tan y = \tan z$.

$$P_p + Q_q = R$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

takin 1st & 2nd   2nd & 3rd

$\tan y\,dx = \tan x\,dy$.

$x\tan y - y\tan x = c_1$

$y\tan z - z\tan y = c_2$.

$\phi[\cdot$

5)

$x^2p + y^2q = z^2$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}$$

taking 1st and second,

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

$$x^{-2}dx = y^{-2}dy$$

$$-\frac{1}{x} = -\frac{1}{y} + c_1$$

$$\frac{1}{y} - \frac{1}{x} = c_1$$

taking 1st and 3rd

$$\frac{dx}{x^2} = \frac{dz}{z^2}$$

$$\frac{1}{2} - \frac{1}{x} = c_2.$$

$$\phi\left(\frac{1}{y} - \frac{1}{x}, \frac{1}{2} - \frac{1}{x}\right] = 0.$$

ϕ being arbitrary Constant.

6) $p - q = \frac{z}{x+y}$

$(x+y)p - q(x+y) = z.$

$$\frac{dx}{x+y} = \frac{dy}{-(x+y)} = \frac{dz}{z}$$

7) $z(x+y)p + z(x-y)q = x^2+y^2$

$$\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2}$$

choosing multipliers

$-x \quad y \quad z$

$$-x\,dx + y\,dy + z\,dz = 0$$

$$-\frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2 = c_1$$

$$y^2 + z^2 - x^2 = c_1$$

Choosing multiplers.

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Solutions

Compatible.

1) $xp - yq = x$ and $x^2p + q = xz$

$f(x, y, pq) = xp - yp - x = 0$

$g(xy, pq) = x^2p + q - xz = 0$

$$[Fg] = \frac{\partial(fg)}{\partial(xp)} + p\frac{\partial[fg]}{\partial[zp]} + \frac{\partial[fg]}{\partial(yq)} + q\frac{\partial[fg]}{\partial(zq)}$$

$$\frac{\partial(fg)}{\partial(xp)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} p-1 & x-y \\ 2xp-z & x^2 \end{vmatrix}$$

$$x^2[p-1) - (2xp-z)(x-y) =$$
$$x^2p - x^2 - (2x^2p - 2xyp - zx + zy)$$
$$x^2p - x^2 - 2x^2p - 2xyp - zx + zy$$
$$= -x^2 - x^2p + 2xyp + zx - zy$$

$$\frac{\partial(fg)}{\partial[zp]} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial z} \end{vmatrix} = \begin{vmatrix} 0 & x-y \\ -x & x^2 \end{vmatrix}$$

$$= -x^2 + xy$$

$$\frac{\partial[fg]}{\partial[yq]} = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} -p & 0 \\ 0 & 1 \end{vmatrix}$$

$$-p$$

$$\frac{\partial[fg]}{\partial[z,q)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ -x & 1 \end{vmatrix}$$

$$\underline{\underline{=0}}$$

$$[Fg] = -x^2 - x^2p - 2xyp - zx + zy] + p[-x^2 + xy]$$
$$+ (-p) + q(0).$$
$$= -x^2 \underline{-x^2p} - 2xyp \underline{-zx} + zy - x^2p + pxy - p.$$

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02/02/26

CHARPIT's METHOD

- This is the general method for Solving first order P.D.E of any degree

Concider $f(x, y, z, P, Q) = 0$ ~~whe~~ we know that $dz = pdx + qdy$

- To use this method we Concider the Charpit's auxilliary equation given by

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \boxed{\frac{dy}{-\frac{\partial f}{\partial q}}}$$

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

Solution Steps

Step I: move all terms to the left to get the form $f(x, y, z, P, Q) = 0$

Step II: Write the Charpit's auxillary equation

Step III obtain values required for the auxilliary equation in step 2 and Substitute into the equation.

Step IV Choose the simplest fraction from step 3 involving atleast P and Q

Step V: The simplest relation in step IV is solved to obtain q and p which can now be used in $dz = pdx + qdy$ which upon integrating gives the required Complete integral.

Example

$$z = xp + yq + p^2 + q^2$$

Solution

Remember $P = \frac{\partial f}{\partial x}$ $q = \frac{\partial f}{\partial y}$.

③ ④

Re-write the given equation.

$$z - xp - yq - p^2 - q^2 = 0$$

Charpits auxilliary equation.

$$\frac{dp}{f_x + pf_z} = \frac{da}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$$\frac{\partial f}{\partial x} = -p, \frac{\partial f}{\partial y} = -q, \frac{\partial f}{\partial z} = 1, \frac{\partial f}{\partial p} = -$$

$$\frac{\partial f}{\partial q} = -y - 2q.$$

Substituting in the auxiliary equation.

$$\frac{dp}{-p+p} = \frac{dq}{-q+q} = \frac{dz}{-p[-x-2p) - q(-y-2q)} = \frac{dx}{x+2p} = \frac{dy}{y+2q}.$$

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{p(x+2p) + q(y+2q)} = \frac{dx}{x+2p} = \frac{dy}{y+2}$$

Choosing the simplest fraction 1st and 2rd

$$\frac{dp}{0} = \frac{dq}{0} \Rightarrow P = \underline{a} \text{ and } q = \underline{b}$$

Since $dp = 0 \rightarrow p = a$, $dq = 0 \rightarrow q = b$ where $\underline{a}$ and $\underline{b}$ are abitrary Constant.

Substituting in the given equation the Complete integral is

$$z = ax + by + a^2 + b^2$$ or we use $dz = pdx + qdy$

$dz = adx + bdy$. intergrating.

$Z = ax + by + c$, a, b, c being arbitrary constants.

TAKE AWAY CAT

1. Form a partial differential equation by eliminating ϕ from

$$L\overset{u}{x}+my+nz=\phi\,\overset{v}{(x^2+y^2+z^2)}$$

Solution

$$\phi(u,v)=0.$$

-2p

$$Lx+my+nz=\phi(x^2+y^2+z^2)$$

Differentiating with respect to x.

~~$L=\phi'(x^2+y^2+z^2)$~~

~~Differentiation with respect to y.~~

~~$m=\phi'(x^2+y^2+z^2)$~~

Differentiat w.r.t x.

$$L+nP=\phi'[x^2+y^2+z^2](2x+2zp)$$

$$\phi'(x^2+y^2+z^2)=\frac{L+P}{2x+2zp}.$$

w.r.t y

$$m+nq=\phi'(x^2+y^2+z^2)(2y+2zq)$$

$$\phi'(x^2+y^2+z^2)=\frac{2y+2zq}{m+nq}$$

$$=\frac{m+nq}{2y+2zq}$$

$$\frac{L+P}{2x+2zp}=\frac{m+nq}{2y+2zq}$$

$$\frac{L+P}{x+zp}=\frac{m+nq}{y+zq}$$

4 $$(L+P)(y+zq)=(x+zp)(m+nq)$$

~~$\partial\theta$~~

$$\frac{\partial\phi}{\partial u}\left[\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right]+\frac{\partial\phi}{\partial v}\left[\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right]=$$

$$\frac{\partial\phi}{\partial u}\left[\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right]+\frac{\partial\phi}{\partial u}\left[\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right]=0,$$

2. Form a partial Differential equation by eliminating the arbitrary functions f and g from $z=yf(x)+xg(y)$

Differentiating w.r.t x

$$P=\frac{dz}{dx}=yf'(x)+g(y) \text{ ---} ✓$$

Differentiating w.r.t y

$$q=\frac{dz}{dy}=f(x)+xg'(y)$$

$$\frac{d^2z}{dxdy}=\boxed{f'(x)+g'(y)}$$

$$f'(x)=\frac{d^2z}{dxdy}-g'(y)$$

$$\frac{dz}{dx}=y\left[\frac{d^2z}{dxdy}-g'(y)\right]+g(y)$$

$$\frac{dz}{dx}=y\frac{d^2z}{dxdy}-yg'(y)+g(y).$$

$$g(y)=\frac{dz}{dx}-yf'(x)$$

$$g(y)=\frac{dz}{dx}-y\left[\frac{d^2z}{dxdy}-g'(y)\right].$$

$$\frac{dz}{dx}=y\frac{d^2z}{dxdy}-yg'(y)+\frac{dz}{dx}-y\frac{d^2z}{dxdy}+yg'(y)$$

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